$$\frac{d^2\mathbf{r}}{dt^2}=\mathbf{u}\times\frac{d\mathbf{r}}{dt}=\mathbf{u}\times(\mathbf{u}\times\mathbf{r})=(\mathbf{u}.\mathbf{r})\mathbf{u}-\mathbf{r},$$

(7)

$$\frac{d^3\mathbf{r}}{dt^3}=\left(\mathbf{u}.\frac{d\mathbf{r}}{dt}\right)\mathbf{u}-\frac{d\mathbf{r}}{dt}=-\frac{d\mathbf{r}}{dt},$$

since $\mathbf{u}.\dfrac{d\mathbf{r}}{dt}=\mathbf{u}.(\mathbf{u}\times\mathbf{r})=0$; and by induction

$$\frac{d^{n+2}\mathbf{r}}{dt^{n+2}}=-\frac{d^n\mathbf{r}}{dt^n} \tag{8}$$

for every $n>0$. Hence Maclaurin's series

$$\mathbf{r}_t=\mathbf{r}_0+\frac{d\mathbf{r}}{dt}.t+\frac{d^2\mathbf{r}}{dt^2}.\frac{t^2}{2\,!}+\ldots$$

reduces to

$$\mathbf{r}_t=\mathbf{r}_0+\left[(\mathbf{u}.\mathbf{r}_0)\mathbf{u}-\mathbf{r}_0\right]\left[\frac{t^2}{2\,!}-\frac{t^4}{4\,!}+\ldots\right]$$

$$+\left[\mathbf{u}\times\mathbf{r}_0\right]\left[\frac{t}{1\,!}-\frac{t^3}{3\,!}+\ldots\right]$$

where the series admit immediate interpretation, giving

$$\mathbf{r}_t=\mathbf{r}_0+[(\mathbf{u}.\mathbf{r}_0)\mathbf{u}-\mathbf{r}_0][1-\cos t]+[\mathbf{u}\times\mathbf{r}_0]\sin t. \tag{9}$$

Here $\mathbf{r}_0$, $\mathbf{r}_t$ are the vectors from the origin to the same point of the body before and after the rotation, respectively, their components being (x, y, z), (x', y', z'); also $\mathbf{u}.\mathbf{r}_0=\lambda x+\mu y+\nu z$, and the components of $\mathbf{u}\times\mathbf{r}_0$ are $(\mu z-\nu y,\ \nu x-\lambda z,\ \lambda y-\mu x)$. Thus equation (9) is equivalent to the Cartesian equations (1) with coefficients expressed as in (3).

2. A Representation of a Solenoidal Vector.

In the general theory of vector functions of a point in space* it is known that a necessary and sufficient condition that a vector have a vanishing curl is that it admit of representation as a potential vector, or one whose components are the partial derivatives of a single scalar function. Symbolically this means that if $\mathbf{v}=\nabla S$, then curl $\mathbf{v}=0$, and conversely, if curl $\mathbf{v}=0$ then there exists a scalar function S such that $\mathbf{v}=\nabla S$. The present note is concerned with an analogous theorem for a solenoidal vector, or one whose divergence vanishes.

Vector Analysis, Chap. III.

It is known, as readily proved by direct differentiation, that if a vector function be the vector product of two potential vectors it is solenoidal; or symbolically, if $\mathbf{v}=\nabla U\times\nabla V$, then $\operatorname{div}\mathbf{v}=0$. The following proof establishes the converse theorem, that if $\operatorname{div}\mathbf{v}=0$ then there exist in general two scalar functions U, V, such that $\mathbf{v}=\nabla U\times\nabla V$.

Let $\mathbf{v}$ be at first any vector function, and u, v two independent integrals of the linear partial differential equation in three independent variables

$$\mathbf{v}.\nabla u=0, \tag{1}$$

which means geometrically that u, v are two functions such that ∇u and ∇v are not collinear, but are both everywhere perpendicular to $\mathbf{v}$. Hence $\mathbf{v}$ can be written

$$\mathbf{v}=w\nabla u\times\nabla v, \tag{2}$$

where w is some scalar function. Then

$$\operatorname{div}\mathbf{v}=w\operatorname{div}(\nabla u\times\nabla v)+\nabla w.(\nabla u\times\nabla v)$$

where the first term vanishes identically so that

$$\operatorname{div}\mathbf{v}=[\nabla u,\ \nabla v,\ \nabla w], \tag{3}$$

this being analytically the Jacobian or functional determinant of the functions u, v, w, with respect to the variables x, y, z, the Cartesian coordinates.

If now $\mathbf{v}$ be solenoidal, or this Jacobian vanish, a well known theorem shows that w must be a function of u, v; so that

$$\mathbf{v}=f(u, v)\nabla u\times\nabla v. \tag{4}$$

Now let U, V be two functions of u, v; then

$$\nabla U\times\nabla V=J\nabla u\times\nabla v \tag{5}$$

where

$$J=\begin{vmatrix} \dfrac{\partial U}{\partial u} & \dfrac{\partial U}{\partial v} \\ \dfrac{\partial V}{\partial u} & \dfrac{\partial V}{\partial v} \end{vmatrix}. \tag{6}$$

It is clearly possible to choose U, V, which are also integrals of (1), so that J, which is a function of u, v, shall be equal to $f(u, v)$; for instance, by taking $U=u$, $V=\int f(u, v)dv$. With any such choice then $\mathbf{v}$ takes the form

(7) $$\mathbf{v} = \nabla U \times \nabla V.$$

A necessary and sufficient condition that a vector function be solenoidal is that it admit of representation as the vector product of two potential vectors.

It should be noted that, when $\mathbf{v}$ is given, U and V are not uniquely determined, so that further conditions may perhaps conveniently be imposed upon them in special cases.

THE POSSIBLE ABSTRACT GROUPS OF THE TEN ORDERS 1909 — 1919.

By DR. G. A. MILLER, University of Illinois.

The real essence of fundamental theorems is frequently exhibited most forcibly by means of illustrative examples, especially when these examples have other elements of interest. The determination of all the possible abstract groups whose orders are equal to the numbers of the ten years 1909—1919 offers numerous instructive illustrations of important theorems, and exhibits some properties of these numbers which are at least of temporary interest.

Since $1909 = 23.83$ is the product of two distinct primes such that the larger diminished by unity is not divisible by the smaller, it results that *the cyclic group of order* 1909 *is the only possible group of this order.* That is, there is only one group whose order is equal to the number of the present year. On the contrary, $1910 = 2.5.191$ is the product of three distinct primes and hence every group of order 1910 contains an invariant subgroup of order 191 and also an invariant subgroup of order 955.* The latter may be either cyclic or non-cyclic, since 191—1 is divisible by 5.

As the group of isomorphisms of the cylic group of order 955 involves three operators of order 2 and the identity, there are four groups of order 1910 which involve a cyclic subgroup of order 955. The group of isomorphisms of the non-cyclic group of order 955 is the holomorph of the group of order 191 and hence it contains only one set of conjugate operators of order 2. When the invariant subgroup of order 955 is non-cyclic an operator of order 2 in the entire group must either transform each operator of this invari nt subgroup into itself or it must transform these operators according to an operator of order 2 in the group of isomorphisms. Hence there are two groups of order 1910 involving a non-cyclic subgroup of the order 955 and *there are exactly six distinct groups of order* 1910; *four of them involve a cyclic subgroup of order* 955 *while the remaining two do not have this property.*

*Cf. Burnside's *Theory of Groups of Finite Order*, 1897, p. 353.

The number $1911=3.7^2.13$ involves four prime factors but these factors are of such a nature that it is comparatively easy to determine all the possible groups of this order. Since 1 is the only divisor of 1911 which is of the form $13k+1$ it results from Sylow's theorem that a group of order 1911 can involve only one subgroup of order 13, and for similar reasons it can involve only one subgroup of order 49. Hence every group of order 1911 contains an invariant abelian subgroup of order 637. There are exactly five groups of order 1911 containing a cyclic subgroup of order 637 since the group of isomorphisms of this cyclic group involves four subgroups of order 3 and the identity. It is not difficult to see that there are ten groups of order 1911 containing the non-cyclic group of order 637, and hence *there are exactly fifteen distinct groups of order* 1911; *two of them are abelian, while the remaining thirteen are non-abelian.*

Every group of order $1912=3^3.239$ contains an invariant subgroup of order 239. If a subgroup of order 8 is also invariant the entire group is the direct product of the group of order 239 and one of the five groups of order 8. Hence there are exactly five groups of order 1912 such that each involves only one subgroup of order 8. If the subgroups of order 8 are not invariant there must be 239 such subgroups having four common operators, since the group of isomorphisms of the group of order 239 does not involve a subgroup of order 4. There are four possible groups in which these common operators form a cyclic group and three in which they form a non-cyclic group. Hence *the total number of abstract groups of order* 1912 *is* 12. Since 1913 is a prime *there is only one group of order* 1913.

The next number, $1914=2.3.11.29$, is the product of distinct primes and hence every group of this order involves an invariant subgroup of each of the orders 29, 29.11, 29.13.3. Each of these invariant subgroups is cyclic* and an operator of order 2 may transform the operators of orders 29, 11, and 3, either into themselves or into their inverses. Hence an operator of order 2 may transform the cyclic subgroup of order 957 in eight distinct ways and *there are exactly eight groups of order* 1914. Since $1915=5.383$ and $383-1$ is not divisible by 5 *there is only one group of order* 1915. *There are only four groups of order* $1916=2^2.479$ since $479-1$ is not divisible by 4.

A group of order $1917=3^3.71$ must be the direct product of the subgroups of order 71 and 27 respectively since $71\equiv 2 \bmod 3$. Hence there are as many distinct groups of order 1917 as there are distinct groups of order 27; that is, *the number of abstract groups of order* 1917 *is* 5. A group of order $1918=2.7.137$ contains an invariant cyclic subgroup of order 7.137 and hence *there are four groups of order* 1918. It is clear that *there is only one group of order* $1919=19.101$. The next number, $1920=2^7.3.5$, lies outside the ten numbers under consideration but it may be remarked that it would be very much more difficult to determine the groups of this one order than it was to determine the groups of the ten orders under consideration. The

Bulletin of the American Mathematical Society, Vol. 5 (1899), p. 235.

year 1920 is so far ahead that it is to be hoped that before it arrives group theory may have made sufficient progress to determine all the groups of this order by means of general theorems. It need scarcely be added that most of the results given above are special cases of general theorems which were not mentioned in every case, since the direct proofs are so evident.

NOTE ON THE GENERAL QUARTIC.

By M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

The general quartic equation

$$\phi(x) \equiv a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0, \tag{1}$$

where the coefficients are all real, can be reduced to the form

$$\phi(x) \equiv ax^4+bx^2+c=0, \tag{2}$$

by the following method.

Suppose the quartic resolved into the factors:

$$(a_1x^2+b_1x+c_1) \text{ and } (a_2x^2+b_2x+c_2).$$

Then effect the rational bilinear transformation $x=\frac{\alpha y+\beta}{y+1}$, obtaining

$$\phi(x) = \left[\frac{a_1(\alpha y+\beta)^2}{(y+1)^2}+\frac{b_1(\alpha y+\beta)}{y+1}+c_1\right]\left[\frac{a_2(\alpha y+\beta)^2}{(y+1)^2}+\frac{b_2(\alpha y+\beta)}{y+1}+c_2\right]=0.$$

In this expression α and β may be chosen so that the coefficients of the first powers of y shall be zero, after clearing of fractions.

The values of α and β fulfilling this condition are easily found to depend upon

$$\alpha\beta=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \tag{3}$$

$$\alpha+\beta=2\frac{a_2c_1-a_1c_2}{a_1b_2-a_2b_1}. \tag{4}$$

That α and β are real, is determined from the equation

$$t^2+2\frac{(a_2c_1-a_1c_2)}{(a_1b_2-a_2b_1)}\,t+\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}=0, \tag{5}$$

by means of the discriminant relation

$$\Delta \equiv \left[\frac{2\,(a_2c_1-a_1c_2)}{(a_1b_2-a_2b_1)}\right]^2-4\left[\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}\right] \tag{6}$$

$$=\frac{4\,[(a_2c_1-a_1c_2)^2-(b_1c_2-b_2c_1)\,(a_1b_2-a_2b_1)]}{(a_1b_2-a_2b_1)^2}.$$

If v_1, w_1, and v_2, w_2 are the two pairs of roots of the quartic, we may assume, without loss of generality, that v_1 is greater than either v_2 or w_2 and w_1 is greater than either v_2 or w_2. From the two quadratic factors whose roots are, respectively, v_1, w_1, and v_2, w_2, we obtain

$$-\frac{b_1}{a_1}=v_1+w_1,\quad \frac{c_1}{a_1}=v_1w_1,\quad -\frac{b_2}{a_2}=v_2+w_2,\text{ and }\frac{c_2}{a_2}=v_2w_2, \tag{7}$$

and these values substituted in the discriminant give for the determinations of its signs,

$$a_1^2a_2^2\,(v_1-v_2)\,(v_1-w_2)\,(w_1-v_2)\,(w_1-w_2) \tag{8}$$

$$=(a_2c_1-a_1c_2)^2-(b_1c_2-b_2c_1)\,(a_1b_2-a_2b_1).$$

If all the roots are real it is evident from (8) that the discriminant is positive. Likewise, if two roots are real and two imaginary, and also if all four roots are imaginary, the discriminant is seen to be positive, the conjugate pairs being, v_1, w_1, and v_2, w_2. Hence, (5) has real roots, and consequently α and β are real.

In the exceptional case where $\frac{b_1}{a_1}=\frac{b_2}{a_2}$, the substitution

$$x=y-\frac{b_1}{2a_1}=y-\frac{b_2}{2a_2},$$

gives directly $\phi(x)=A_1y^4+A_2y^2+A_3$, where A_1, A_2, A_3 are real constants.

Finally, from (3), (4) and (7),

$$\alpha\beta=\frac{(v_2+w_2)v_1w_1-v_2w_2(v_1+w_1)}{(v_1+w_1)-(v_2+w_2)}, \tag{9}$$

(10) $$\text{and } \alpha+\beta=\frac{2v_1w_1-2v_2w_2}{(v_1+w_1)-(v_2+w_2)}.$$

From these equations it is easily seen that

$$2\alpha\beta-(\alpha+\beta)(v_1+w_1)=-2v_1w_1,$$
$$\text{and } 2\alpha\beta-(\alpha+\beta)(v_2+w_2)=-2v_2w_2.$$

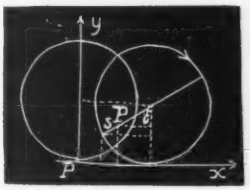
But these are the characteristic equations for harmonic pairs. Therefore, the pair α, β is harmonic with respect to the pairs v_1, w_1 and v_2, w_2.

ANOTHER WAY TO GENERATE THE CYCLOID.

By H. SCHAPPER, Fayetteville, Ark.

Consider a point describing a simple harmonic motion along the diameter of a uniformly rolling circle, and let both motions begin simultaneously. If r be the radius of the circle, and ϵ the angle of rolling, and if a complete unrolling of the circle correspond to a full period of the simple harmonic motion, then we get

$$s=r(1-\cos\epsilon)$$



as the equation for the simple harmonic motion.

From the figure we see that

$$x=r(\epsilon-\sin\epsilon)+r(1-\cos\epsilon)\sin\epsilon,$$
$$y=r(1-\cos\epsilon)+r(1-\cos\epsilon)\cos\epsilon.$$

After an easy reduction we get

$$x=r(\epsilon-\tfrac{1}{2}\sin 2\epsilon), \quad y=r\sin^2\epsilon.$$

These equations may be written in the following way, or

$$x=\frac{r}{2}(2\epsilon-\sin 2\epsilon), \quad y=\frac{r}{2}(1-\cos 2\epsilon),$$

or

$$x=R(\phi-\sin\phi), \quad y=R(1-\cos\phi).$$

which is the parametric form of the equations of the cycloid. The cycloid thus generated is the same as if described by a simple rolling of a circle of radius $\frac{1}{2}r$. From

$$\frac{dy}{dx}=\frac{\sin 2\varepsilon}{(1-\cos 2\varepsilon)}=\cot\varepsilon=\tan(\tfrac{1}{2}\pi-\varepsilon),$$

it follows that the diameter $2r$ of the circle is always tangent to the curve, so that we get the points of the curve as well as their corresponding tangents at the same time, which is rather an advantage. The construction of the cycloid is thus simplified. Besides, it is of interest that the definition of the cycloid is thus given in a different way from the one commonly employed.

DEPARTMENTS.

PROBLEMS FOR SOLUTION.

ALGEBRA.

305. Proposed by S. A. COREY, Hiteman, Iowa.

Prove or disprove, that $\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^2+4}=\frac{\pi}{4}-\frac{\pi}{8}\left(\frac{\cosh\pi}{\sinh\pi}\right)$.

Solution by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Taking the logarithm of the well known formula

$$\cos\tfrac{1}{2}\pi x=\left(1-\frac{x^2}{1}\right)\left(1-\frac{x^2}{9}\right)\left(1-\frac{x^2}{25}\right)\ldots$$

and differentiating, we have

$$\tfrac{1}{2}\pi\tan\tfrac{1}{2}\pi x=\frac{2x}{1-x^2}+\frac{2x}{9-x^2}+\frac{2x}{25-x^2}+\ldots$$

Putting $x\sqrt{-1}$ in place of x, we have

$$\tfrac{1}{2}\pi\tan(\tfrac{1}{2}\pi x\sqrt{-1})=2x\sqrt{-1}\left[\frac{1}{1+x^2}+\frac{1}{9+x^2}+\frac{1}{25+x^2}+\ldots\right]$$

Putting $x=2$, we have

$$\tfrac{1}{2}\pi\tan(\pi\sqrt{-1})=4\sqrt{-1}\left[\frac{1}{1+4}+\frac{1}{9+4}+\frac{1}{25+4}+\ldots\right]$$

But, $\tan(\pi\sqrt{-1})=\dfrac{\tanh\pi}{\sqrt{-1}}$; $\therefore\dfrac{1}{1+4}+\dfrac{1}{9+4}+\dfrac{1}{25+4}+\ldots=\tfrac{1}{8}\pi\tanh\pi$.

Also solved by G. B. M. Zerr, and V. M. Spunar.

306. Proposed by J. C. CORBIN, Pine Bluff, Ark.

Muir gives the following problem:

Prove: $\begin{vmatrix}1 & a & a & a^2\\ 1 & b & b & b^2\\ 1 & c & c' & cc'\\ 1 & d & d' & dd'\end{vmatrix}=(a-b)\begin{vmatrix}1 & ab & a+b\\ 1 & cd' & c+d'\\ 1 & c'd & c'+d\end{vmatrix}$

which, of course, can be solved by finding the terms of both determinants. Is there any method of changing from one form to the other which is direct?

Solution by J. A. CAPARO, C. E., Notre Dame University, Notre Dame, Ind.

Subtracting the first row from each one of the others:

$$\begin{vmatrix}1 & a & a & a^2\\ 0 & b-a & b-a & b^2-a^2\\ 0 & c-a & c'-a & cc'-a^2\\ 0 & d-a & d'-a & dd'-a^2\end{vmatrix}=(a-b)\begin{vmatrix}1 & 1 & b+a\\ d-a & d'-a & dd'-a^2\\ c-a & c'-a & cc'-a^2\end{vmatrix}=\triangle.$$

Multiply the first row by a and add it to the next two rows.

$$\triangle=(a-b)\begin{vmatrix}1 & 1 & b+a\\ d & d' & dd'+ba\\ c & c' & cc'+ba\end{vmatrix}=$$

$(a-b)[d'cc'+d'ba-c'dd'-c'ba-dcc'-dab+cdd'+abc+bdc'+adc'-bcd'-acd']$ which is the same as $[ab(c+d'-c'-d)+cd'(c'+d-a-b)-c'd(d'+c-a-b)]\times(ab)$ or $\triangle=(a-b)[ab(c+d')-ab(c'+d)+cd'(c+d)-cd'(a+b)-c'd[d'+c)+c'd(a+b)]$, which can be written

$$\triangle=(a-b)\left[\begin{vmatrix}ab & a+b\\ cd' & c+d'\end{vmatrix}-\begin{vmatrix}ab & a+b\\ c'd & c'+d\end{vmatrix}+\begin{vmatrix}cd' & c+d'\\ c'd & c'+d\end{vmatrix}\right]$$

which by Cor. 1, page 34, of Hanus' *Elements of Determinants*, is

$$\triangle=(a-b)\begin{vmatrix}1 & ab & a+b\\ 1 & cd' & c+d'\\ 1 & c'd & c'+d\end{vmatrix}.$$

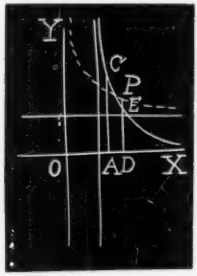
Also solved by J. W. Clawson, and G. B. M. Zerr.

307. Proposed by J. SCHEFFER, Hagerstown, Md.

If $y^x=2$ and $x^y=3$, find x and y.

I. Solution by E. B. ESCOTT, Ann Arbor, Mich.

The solution (x, y) of these two equations, is the same as the intersection of the curves (1) and (2).



In the figure, (1) is drawn with full lines, (2) with dotted lines. We see that outside of (0, 0) there is but one point of intersection, P. If we cut the curves by a line $x=c$, it is evident that the value of y from (2) is nearer to the required value than that from (1). If we cut the curve by the line $y=a$, the value of x from (1) is nearer to the required value than the value of x from (2). A rapid approximation can then be found by taking any value of $x>1$ and solving (2) for y. With this value of y substituted in (1), solve for x, etc. If we do this for two values of x, one such that ABC is to the left of P and another such that DEF is to the right of P, we shall approach the point P from both directions and will have an upper and lower limit for the coordinates of P at each step. In this way I find the approximation

$$2.2390<x<2.23939, \quad 1.3627<y<1.36285.$$

Solved similarly by S. Lefseletz.

II. Solution by FRANCIS RUST, C. E., Allegheny, Pa.

The problem proposed by Mr. Scheffer is found in Heis' *Sammlung von Beispielen und Aufgaben, etc.,*" §106, No. 12.

$x^y=b=3$, $y^x=a=2$ makes $y=\dfrac{\log b}{\log x}$, $x=\dfrac{\log a}{\log y}$; *i. e.*, $x\log y=a$, from which is derived the transcendental equation:

$$x\log\log b-x\log\log x-\log a=0=f(x).$$

Solving same by means of Newton's method:

$$h=-\frac{f(x)}{f'(x)}; \quad f'(x)=\log\log b-\log\log x-\frac{(\log e)^2}{\log x}.$$

$a=2$, $b=3$, $\log e=0.4342945$, results: $x=2.2392498$, $y=1.3628042$.

Also solved by V. M. Spunar, G. B. M. Zerr, J. A. Caparo, A. H. Holmes, and the Proposer.

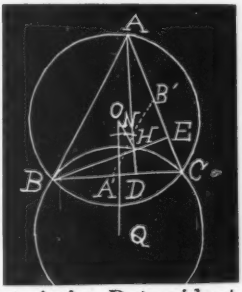
GEOMETRY.

338. Proposed by C. N. SCHMALL, 239 East 7th Street, New York.

Given the base and vertical angle of a triangle, find the locus of the center of its nine-point circle.

I. Solution by J. W. CLAWSON, Ursinus College, Collegeville, Pa.

(1) Take an angle BAC equal to the given vertical angle and from any point B in one arm of it describe an arc with BC equal to the given base as radius cutting the other arm in C. Draw a circle through A, B, C. Then BC being fixed, the circle ABC is the locus of the vertex A.



(2) Find the center O of this circle, draw OA' the perpendicular bisector of BC, and OB' the perpendicular bisector of CA. Drop AD perpendicular to BC, and BE perpendicular to CA, intersecting AD in H.

The nine-point circle passes through A', D and B', E. Therefore the lines bisecting $A'D$ and $E'E$ at right angles intersect in N, the center of the nine-point circle. But evidently each of these lines bisects OH. Therefore N bisects OH.

(3) Join $A'B'$. The triangles AHB and $OA'B'$ are similar, having their sides mutually parallel. Also $AB=2A'B'$. So $AH=2OA'$, and is therefore constant.

(4) Since AH is constant in length, and as A moves AH moves parallel to itself, the locus of H is a circle of radius equal to OA' and whose center, Q, lies on OA' produced so that $OQ=AH=2OA'$. Draw this circle.

(5) Since N bisects OH, O is a fixed point, and the locus of H is a circle, the locus of N is a circle, whose center bisects OQ, and is therefore the point A', and whose radius is half the radius of the original circle.

Also solved by J. M. Meyer, S. J,. Francis Rust, V. M. Spunar, G. B. M. Zerr, and J. Scheffer.

CALCULUS.

265. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Find two curves which possess the property that the tangents TP and TQ to the inner one always make equal angles with the tangent TT' to the outer.

II. Solution by F. H. SAFFORD, Ph. D., University of Pennsylvania.

Let $y=\phi(x)$ be the equation of the inner curve, and let the coordinates of P and Q, points on this curve, be, respectively, (x_1, y_1), (x_2, y_2). Assuming that the outer curve exists, let its equation be $y=f(x)$, and let $f'(x)=\tan \alpha$.

The two tangent lines from P and Q, respectively, are, if $\phi'(x)=\tan \theta$,

$$y\cos\theta_1 - x\sin\theta_1 + x_1\sin\theta_1 - y_1\cos\theta_1 = 0,$$
$$y\cos\theta_2 - x\sin\theta_2 + x_2\sin\theta_2 - y_2\cos\theta_2 = 0.$$

while the bisector of the angle between these two lines is

$$y(\cos\theta_1 \pm \cos\theta_2) - x(\sin\theta_1 \pm \sin\theta_2)$$
$$+ x_1\sin\theta_1 - y_1\cos\theta_1 \pm (x_2\sin\theta_2 - y_2\cos\theta_2) = 0.$$

Let $(\bar{x}, \bar{y})$ be T, the intersection of the tangents;

$$\therefore \bar{x} = \frac{x_2\phi'(x_2) - x_1\phi'(x_1) + \phi(x_1) - \phi(x_2)}{\phi'(x_2) - \phi'(x_1)},$$

$$\bar{y} = \frac{\phi(x_1)\phi'(x_1) - \phi(x_2)\phi'(x_2) + (x_2 - x_1)\phi'(x_1)\phi'(x_2)}{\phi'(x_2) - \phi'(x_1)}.$$

The normal to the outer curve at T is

$$y\sin\bar{\alpha} + x\cos\bar{\alpha} - \bar{x}\cos\bar{\alpha} - \bar{y}\sin\bar{\alpha} = 0.$$

By the assumption in the given problem, the bisector and normal written above must be identical, hence by comparison of coefficients,

$$\frac{\cos\theta_1 \pm \cos\theta_2}{\sin\bar{\alpha}} = -\frac{\sin\theta_1 \pm \sin\theta_2}{\cos\bar{\alpha}} = \frac{x_1\sin\theta_1 - y_1\cos\theta_1 \pm (x_2\sin\theta_2 - y_2\cos\theta_2)}{-(\bar{x}\cos\bar{\alpha} + \bar{y}\sin\bar{\alpha})}.$$

From the first two members of the last equation, $\tan\bar{\alpha} = \pm\left[\tan\frac{\theta_1 + \theta_2}{2}\right]^{\mp 1}$, but since $\tan\bar{\alpha} = f'(\bar{x})$, it follows that $\tan(\theta_1 + \theta_2) \equiv \frac{\phi'(x_1) + \phi'(x_2)}{1 - \phi'(x_1)\phi'(x_2)}$ should be a function of $\bar{x}$. The condition that one is a function of the other is

$$\frac{\partial}{\partial x_1}\tan(\theta_1 + \theta_2)\frac{\partial\bar{x}}{\partial x_2} - \frac{\partial}{\partial x_2}\tan(\theta_1 + \theta_2)\frac{\partial\bar{x}}{\partial x_1} = 0.$$

This leads to a relation between x_1 and x_2, and if $\bar{y}$ is to be a function of $\bar{x}$, a similar argument leads to another relation between x_1 and x_2, not necessarily the same as the former. As the original problem assumed no relation whatever between x_1 and x_2, it appears that the problem is indeterminate.

III. Solution by E. B. ESCOTT, Ann Arbor, Mich.

The question is indefinite. Where the inner curve is a conic section, the problem may be solved as follows.

(a) Inner curve an ellipse,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0.$$

Let (x_1, y_1) be the point T. Polar of (x_1, y_1) with respect to the conic, is

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1.$$

The curve

$$k\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)-l\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2=0,$$

is a conic tangent to the given ellipse at points of intersection with the polar of T (Charles Smith, *Conic Sections*, p. 203). If this passes through T, it will be the equation of the tangents from T to the conic.

Putting $x=x_1$, $y=y_1$, we find that

$$k=l\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right).$$

The equation of the tangents from T is, therefore

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)-\left(\frac{x_1x}{a^2}+\frac{y_1y}{b^2}-1\right)^2=0.$$

The slopes of the lines are determined by the terms of the second degree, which are

$$(b^2-y_1^2)x^2+2x_1y_1xy+(a^2-x_1^2)y^2.$$

The lines bisecting the angles between these lines will be parallel to the lines whose equation is

$$\frac{x^2-y^2}{b^2-y_1^2-a^2+x_1^2}=\frac{xy}{x_1y_1},$$

(Charles Smith, *Conic Sections*, p. 34).

The slope y/x of these lines is determined from

$$\left(\frac{y}{x}\right)^2+\frac{b^2-y_1^2-a^2+x_1^2}{x_1y_1}\left(\frac{y}{x}\right)-1=0.$$

Therefore, the differential equation of the curve of which these lines are tangents is

$$\left(\frac{dy}{dx}\right)^2+\frac{b^2-y^2-a^2+x^2}{xy}\left(\frac{dy}{dx}\right)-1=0.$$

The solution of this equation is the conic section

$$\frac{x^2}{m^2}+\frac{y^2}{n^2}=1,$$

or, in parameter form, $x=m\cos\phi$, $y=n\cos\phi$. From these we get

$$\frac{dy}{dx}=-\frac{n}{m}\cot\phi.$$

Substituting these values of x, y, and dy/dx in the differential equation, we get $m^2-n^2=a^2-b^2$.

Therefore, the required outer curve is any one of a system of ellipses or hyperbolas confocal with the inner curve.

(b) If the inner curve is an hyperbola, we have the same result.

(c) If the inner curve is the parabola, $y^2=4p(x+p)$.

Proceeding in the same way, the equation of the tangents is

$$(y^2-4px-4p^2)(y'^2-4px'-4p^2)-[yy'-2p(x+x')-4p^2]=0.$$

The differential equation is

$$\left(\frac{dy}{dx}\right)^2+\frac{2x}{y}\left(\frac{dy}{dx}\right)-1=0.$$

The solution is, $y^2=4c(x+c)$.

This is a system of parabolas confocal with the given one.

Also solved by the Proposer.

MECHANICS.

222. Proposed by W. J. GREENSTREET, Stroud, England.

Find the maximum angle of inclination to the line of greatest slope of a uniform rod resting on a rough inclined plane and capable of turning freely round a point on it.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

By the maximum angle is evidently meant the angle of limiting equi-

librium. Let R=the reaction of the rod concentrated at G, its center of gravity; μ=coefficient of friction; W=weight of rod; β=inclination of plane to horizon; θ=required angle.

The reaction R at G is perpendicular to the plane, and is $R=W\cos\beta$; while $W\sin\beta$ is the pull down the plane. Now as friction acts opposite to the direction of motion, we get $W\sin\beta\sin\theta=\mu R$.

$\therefore W\sin\beta\sin\theta=\mu W\cos\beta$, or $\sin\theta=\mu\cot\beta$; $\theta=\sin^{-1}(\mu\cot\beta)$.

223. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A sphere, radius $r=\frac{1}{3}$ inches, density $\delta=11.38$, falls from a height $h=500$ feet, into a lake depth $l=40$ feet. Find time of falling to surface of lake, time of falling from surface of lake to bottom, and total time of falling. Also the velocity at the bottom.

Solution by the PROPOSER.

Let $r=\frac{1}{3}$ inch $=\frac{1}{36}$ feet=radius of ball; $g=32.2$=gravity; $e_2=g$=weight of unit volume of water; δ=density of air=.001293; $e_1=g\delta$=weight of unit volume of air; R_1v^2=resistance of air; R_2v^2=resistance of water; A=greatest sectional area of sphere=πr^2; k=a constant=0.51 for the sphere; m=mass of sphere=$\frac{1}{16}g$; v_1=velocity at surface of lake; v_2=velocity at bottom of lake. Then the equations of motion are

$$m\frac{dv}{dt}=mg-Rv^2\ldots(1), \qquad mv\frac{dv}{dx}=mg-Rv^2\ldots(2).$$

From (1), $t=m\int\frac{dv}{mg-Rv^2}$. For the time, T, from rest to the surface of the lake, $R=R_1$, and the limits of v are 0 and v_1.

$$\therefore T=\tfrac{1}{2}\sqrt{\frac{m}{gR_1}}\log\left[\frac{\sqrt{(mg)}+\sqrt{(R_1)}v_1}{\sqrt{(mg)}-\sqrt{(R_1)}v_1}\right]\ldots(3).$$

For the time, T_1, from the surface to the bottom of the lake, $R=R_2$, and the limits of v are v_1 and v_2.

$$\therefore T_2=\tfrac{1}{2}\sqrt{\frac{m}{gR_2}}\log\left[\frac{[\sqrt{(mg)}+\sqrt{(R_2)}v_2][\sqrt{(mg)}-\sqrt{(R_2)}v_1]}{[\sqrt{(mg)}-\sqrt{(R_2)}v_2][\sqrt{(mg)}+\sqrt{(R_2)}v_1]}\right]\ldots(4).$$

From (2), $x=m\int\frac{vdv}{mg-Rv^2}$.

Between the limits v and V we get for x,

$$x=\frac{m}{2R}\log\left[\frac{mg-RV^2}{mg-Rv^2}\right].$$

$$\therefore v^2=\frac{mg}{R}(1-e^{-(2Rx/m)})+V^2e^{-(2Rx/m)}.$$

For the velocity at the surface of the lake, $R=R_1$, $V=0$, $x=h$, $v=v_1$.

$$\therefore v_1^2=\frac{mg}{R_1}(1-e^{-(2h/m)})\ldots(5).$$

For velocity at bottom of lake, $x=l$, $V=v_1$, $R=R_2$, $v=v_2$.

$$\therefore v_2^2=\frac{mg}{R_2}(1-e^{-(2l/m)})+v_1^2e^{-(2l/m)})\ldots(6).$$

Following Rankin: $R_1v^2=\frac{kAe_1v^2}{2g}$, $R_2v^2=\frac{kAe_2v^2}{2g}$.

$$\therefore R_1=\frac{0.51\,\pi r^2 g\,\delta}{2g}=.0000008;\quad R_2=\frac{0.51\,\pi r^2 g}{2g}=.000618.$$

Since $h=500$, $l=40$, we get: from (5), $v_1=162.447$ feet; from (6), $v_2=10.0564$ feet; from (3), $T=5.814$ seconds; from (4), $T_1=1.773$ seconds. $T+T_1=7.587$ seconds=total time. These results would be slightly changed for different values of R_1, R_2.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

NOTE ON PROBLEM 152.

On referring to the memorandum book from which the problem was taken, I find that p is prime, for which case the theorem would appear to be true. PROPOSER.

155. Proposed by R. D. CARMICHAEL, Anniston, Ala.

If p and q are primes and m and n are any integers, find the cases in which the equation $p^m-q^n=1$ may be satisfied.

Solution by the PROPOSER.

Some values, found by inspection, are given by Zerr in the MONTHLY for December, 1908. A complete solution may be effected by aid of the following lemma (see *Annals of Mathematics*, Vol. 8, No. 4, p. 15).

If x is a positive integer >1, x^t-1 has a prime factor not dividing x^u-1 $(u<t)$, except in the cases $t=2$, $x=2^v-1$, $v\geqq 2$; $t=6$, $x=2$. Such prime

factors of x^t-1 *are of the form* $st+1$, *and evidently if* t *is odd and* >1 *they are of the form* $2st+1$. Such a prime is called a characteristic factor.

Now, either p or q is even. Suppose that q is even. Then we have

$$2^n=p^m-1.$$

Excluding for the moment the cases of exception in the lemma, we must have $m=1$, since otherwise p^m-1 would have no factor of the form $sm+1$. This gives the solution,

(1) $$m=1;\ q=2;\ p=2^n+1;$$

where n has any value making 2^n+1 prime. The exceptional case in the present instance is $p=2^v-1$, a prime, and $m=2$. This gives

$$2^n=(2^v-1)^2-1=2^{2v}-2^{v+1}=2^{v+1}(2^{v-1}-1).$$

Hence, $v=2$. Therefore, we have the solution

(2) $$m=2;\ q=2;\ n=3;\ p=3.$$

Next, suppose that p is even. Then we have

$$2^m=q^n+1.$$

Hence 2 is the characteristic prime factor of $q^{2n}-1=(q^n+1)(q^n-1)$, if it has such characteristic factor. Such factor must be of the form $2sn+1\neq 2$. Hence, we now have no solution except for an exceptional case of the lemma. The only such case applying to the form $q^{2n}-1$ is that for $n=1$, $q=2^v-1$, a prime. This gives $m=v$. Hence, we have the following solution,

(3) $$n=1;\ q=2^m-1,\text{ a prime};\ p=2;$$

m=any number making 2^m-1 a prime. Hence, all the solutions of the problem are included in (1), (2), and (3).

Remarks by E. B. ESCOTT, Ann Arbor, Mich.

Gerono, *Nouvelles Annales*, 1870, p. 469; 1871, p. 204, has given a demonstration for the case where one of the roots is prime. Catalan says that he wasted a year trying to find a proof. The question was proposed in *L'Intermédiaire des Mathématiciens* as question 487, and the special case $x^2-1=y^3$ as question 664. Father Pepin proved that the last case is impossible except for $x=3$, $y=2$ (*L'Intermédiaire des Mathématiciens*, 1896, pp. 284-5) by using a theorem regarding the impossibility of the equation

$$u^3+x^3=2y^3,$$

which is proven in Legendre, *Théorie des Nombres*, p. 347, and in Euler's *Algebra*, Chap. XV, Part 2. He remarks that neither Euler nor Legendre have justified the use they make of the complex numbers $p+q\sqrt{-3}$. See also, *Crelle*, Vol. 27 (1844), p. 192.

PROBLEMS FOR SOLUTION.

ALGEBRA.

311. **Proposed by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y.**

Find, by Cardan's Method, the real root (4) of $x^3-6x^2+10x=8$.

312. **Proposed by J. A. CAPARO, C. E., Notre Dame University, Notre Dame, Ind.**

Two roots of the cubic $x^3-px^2+qx-c=0$ are equal. Find their value in terms of p, q, and c.

GEOMETRY.

342. **Proposed by G. I. HOPKINS, M. A., Instructor in Mathematics and Astronomy, Manchester, N. H.**

Given, circle DEF inscribed in triangle ABC and circumscribing the triangle DEF, D, E, F being the points of contact; AH is drawn through center, N, meeting chord DF in H. Through H is drawn BK meeting AC in K. Prove triangle ABK isosceles.

CALCULUS.

269. **Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

Prove that $\int_0^1 (x^a+x^{-a})\log\left(\frac{1+x}{1-x}\right)\frac{dx}{x}=\frac{\pi}{a}\tan(\tfrac{1}{2}\pi a)$.

270. **Proposed by S. A. COREY, Hiteman, Iowa.**

Prove that $\sum_{x=0}^{x=\infty}\frac{1}{(a^2+x^2)^n}=\frac{\pi}{2a^{2n-1}}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{(2n-3)}{(2n-2)}+\frac{1}{2a^{2n}}$, n being a positive integer >1.

MECHANICS.

225. **Proposed by W. A. BALDWIN, Senior in Drury College, Springfield, Mo.**

Find, *by means of polar coordinates*, the moment of inertia about the origin of the area between the parabola $ay=2(a^2-x^2)$, the circle $x^2+y^2=a^2$, and the axis of Y.

226. **Proposed by W. J. GREENSTREET, M. A., Stroud, England.**

A frustum of a cone, vertical angle α, is cut off by two spheres whose centers are the vertex. The radius of one sphere is n times that of the other, and the density of the cone varies as the distance of the vertex. Find the ratio into which the centroid of the frustum divides the axis.

3

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ON THE THEORY OF FUNCTIONS OF A TRIPLE VARIABLE.*

By R. D. CARMICHAEL, Anniston, Alabama.

§1. The Algebra Defined.

Since single real numbers may be represented by the points of a linear continuum we may speak of their algebra as that of one-dimensional extent. Similarly the algebra of double numbers (or number pairs or complex numbers) finds its geometrical image in the points of a surface, or a two-dimensional extent. But no attempt has ever succeeded in defining a system of triple numbers whose algebra is entirely analogous to that of one-dimensional and two-dimensional extent. In fact it has been shown† that no triple algebra exists at the same time conserving all the formal laws of the fundamental operations as developed in our ordinary algebra and also maintaining the important theorem: *If a product of two factors equals zero one at least of the factors is zero.* This theorem plays a fundamental rôle in algebraic analysis and no complete surrender of it can be permitted in constructing a new algebra.

Nevertheless, in attempting to devise a system of units such that numbers built up from them may be imaged in three-dimensional extent, it is just this law which I have agreed to surrender. The problem is then to find a system of units such that the additional complication incident to the surrender of this law shall be reduced to a minimum. Several systems of units with their laws were assumed arbitrarily, but their algebra in every case presented almost insurmountable difficulties. Finally by an analysis of commutative operations in space—translations and rotations—the system of units here employed was defined, first geometrically and then algebraically. We are now concerned with the latter alone.

We begin by defining a system of units by the following equations:

$$\text{I.}\quad r.r=r,\ \frac{r}{r}=r,\ ri=ir=i,\ i^2+r=0,\ s.s=s,\ \frac{s}{s}=s,\ r.s=s.r=i.s=s.i=0.$$

*Read before the American Mathematical Society, October 31, 1908.
†See Stolz and Gmeiner's *Theoretische Arithmetik*.